

# BRUHAT CELLS IN THE NILPOTENT VARIETY AND THE INTERSECTION RINGS OF SCHUBERT VARIETIES

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## 1. Introduction

Let  $G$  be a complex semisimple Lie group with fixed opposite Borel subgroups  $B$  and  $B^-$ , and let  $H$  be the maximal torus  $B \cap B^-$ .  $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$  denote the Lie algebras of  $G$ ,  $B$ ,  $H$  respectively and  $W = N(H)/H$  is the Weyl group of  $(G, H)$ . A famous result in Lie theory says that the cohomology algebra  $H^*(G/B; \mathbb{C})$  of the flag variety  $G/B$  of  $G$  is isomorphic to the coordinate ring  $A(\mathcal{N} \cap \mathfrak{h})$  of the scheme-theoretic intersection of the nilpotent variety  $\mathcal{N} \subset \mathfrak{g}$  and the Cartan subalgebra  $\mathfrak{h}$ . The purpose of this paper is to extend this result to Schubert varieties  $X_w := \overline{BwB/B}$  in  $G/B$ , where  $w \in W$ .

We introduce a locally closed stratification  $\mathcal{B}_w$  of  $\mathcal{N}$  by "Bruhat cells" defined by putting  $\mathcal{B}_w = \text{Ad}(Bw^{-1}B)u$ , where  $u$  is the nilradical of  $\mathfrak{b}$ .  $\mathcal{N}_w := \overline{\mathcal{B}_w}$  is a Zariski closed irreducible cone in  $\mathfrak{g}$  such that  $\mathcal{N}_w \subseteq \mathcal{N}_y$  if and only if  $X_w \subseteq X_y$ . Recall that the scheme-theoretic intersection of varieties  $Z_1$  and  $Z_2$  in  $\mathfrak{g}$  is the scheme  $Z_1 \cap Z_2$  defined by the ideal  $I(Z_1) + I(Z_2)$  where  $I(Z_i)$  is the ideal of  $Z_i$  in the coordinate  $A(\mathfrak{g})$  of  $\mathfrak{g}$ . By definition, the coordinate ring  $A(Z_1 \cap Z_2)$  of  $Z_1 \cap Z_2$  is  $A(\mathfrak{g}) / (I(Z_1) + I(Z_2))$ . We will prove

**Theorem 1.** *For each  $w \in W$ , there exists a surjective degree doubling homomorphism of graded  $\mathbb{C}$ -algebras  $\psi_w : A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow H^*(X_w; \mathbb{C})$  such that if  $X_w \subseteq X_y$ , the diagram*

$$(1.1) \quad \begin{array}{ccc} A(\mathcal{N}_y \cap \mathfrak{h}) & \xrightarrow{\psi_y} & H^*(X_y; \mathbb{C}) \\ \downarrow & & \downarrow \\ A(\mathcal{N}_w \cap \mathfrak{h}) & \xrightarrow{\psi_w} & H^*(X_w; \mathbb{C}) \end{array}$$

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Received April 30, 1990 and, in revised form, December 4, 1991. The author's work was partially supported by a grant from the National Sciences and Engineering Research Council of Canada.

commutes, where the vertical maps are induced by the natural inclusions. If  $X_w$  is smooth, then  $\psi_w$  is an isomorphism.

We remark that if  $w_0$  is the longest element of  $W$ , then  $\mathcal{N}_{w_0} = \mathcal{N}$  and  $\psi_{w_0}$  is the classical isomorphism. The homomorphisms  $\psi_w$  are constructed by relating  $\mathcal{N}_w \cap H$  and the zero scheme of the algebraic vector field  $V_e$  on  $G/B$  studied in [1], where  $e$  is a homogeneous principal nilpotent in  $\mathfrak{b}$  (see §2).  $V_e$  has exactly one zero, the coordinate ring  $A(Z_e)$  of the zero scheme  $Z_e$  of  $V_e$  is [known to be] a graded  $\mathbb{C}$ -algebra, and there exists an isomorphism of graded  $\mathbb{C}$ -algebras  $\alpha: A(Z_e) \rightarrow H^*(G/B; \mathbb{C})$ . Moreover,  $V_e$  is tangent to  $X_w$  at all smooth points, so one can consider the scheme-theoretic intersection  $Z_e \cap X_w$ . The coordinate ring  $A(Z_e \cap X_w)$  is also graded, and an application of a result in [3] gives the existence of a surjective graded  $\mathbb{C}$ -algebra morphism  $\alpha_w: A(Z_e \cap X_w) \rightarrow H^*(X_w; \mathbb{C})$  such that the analog of diagram (1.1) commutes when  $X_w \subset X_y$  and such that  $\alpha_w$  is an isomorphism if  $X_w$  is smooth.

The key to proving Theorem 1 is thus to produce isomorphisms  $\beta_w: A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow A(Z_e \cap X_w)$  having the usual naturality properties. To do so, we consider the morphism  $\phi_e: U^- \rightarrow \mathfrak{g}$  given by  $\phi_e(u) = \text{Ad}(u^{-1})e$ ,  $U^-$  being the unipotent radical of  $B^-$ . Letting  $\rho: U^- \rightarrow G/B$  be the isomorphism  $\rho(u) = u \cdot B$  of  $U^-$  onto the open cell  $U$  centered at  $B$  and noting that  $\rho^{-1}(X_w \cap U) = \phi_e^{-1}(\mathcal{N}_w)$ , we can state

**Theorem 2.** *The comorphism  $(\phi_e \rho^{-1})^*: A(\mathfrak{g}) \rightarrow A(U)$  induces, for each  $w \in W$ , a degree-doubling isomorphism  $\beta_w: A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow A(Z_e \cap X_w)$  so that if  $X_w \subset X_y$ , then  $\beta_w$  and  $\beta_y$  commute with the natural restrictions.*

It has been an open question whether the homomorphisms  $\alpha_w: A(Z_e \cap X_w) \rightarrow H^*(X_w; \mathbb{C})$  are isomorphisms. Recently this question has been partially answered by the following two results.

**Theorem [5].** *If  $G = \text{SL}_n(\mathbb{C})$ , every  $\alpha_w$  is an isomorphism.*

On the other hand, Dale Peterson has shown

**Theorem 3.** *Suppose  $\omega$  is a nonminuscule fundamental dominant weight for  $\mathfrak{h}$  and let  $r \in W$  be the reflection corresponding to the simple root associated to  $\omega$ . Let  $w = w_0 r$ , where  $w_0$  is  $W$ 's longest element. Then  $\dim_{\mathbb{C}} A(Z_e \cap X_w) = \dim_{\mathbb{C}} A(Z_e)$ , and consequently  $\alpha_w$  is not injective.*

Theorem 3 is proved in the Appendix. Note that  $\text{SL}_n(\mathbb{C})$  is the only simple group in which all fundamental dominant weights are minuscule. Hence for  $G$  simple not of type  $A_n$  there exist codimension-one  $X_w$  for which  $\alpha_w$  is not an isomorphism.

As a consequence of the fact that all  $\alpha_w$  are isomorphisms if  $G = \mathrm{SL}_n(\mathbb{C})$ , one obtains the

**Corollary.** *For  $G = \mathrm{SL}_n(\mathbb{C})$ , all  $\psi_w : A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow H^*(X_w; \mathbb{C})$  are isomorphisms.*

There is a conjectured definition of  $\psi_w$  not involving  $A(Z_e \cap X_w)$  which we discuss since it yields information on when  $\alpha_w$  is an isomorphism. Let  $t \in \mathfrak{h}$  and set  $\mathcal{B}_{w,t} := \mathrm{Ad}(Bw^{-1}B)t$ . Using a result of [5], we show in Theorem 8 that  $H^*(X_w; \mathbb{C})$  is isomorphic with the graded ring  $A(\mathfrak{g})/\mathrm{gr}(I(\mathcal{B}_{w,t}) + I(\mathfrak{h}))$ . Here  $\mathrm{gr}I$  denotes the ideal generated by the leading terms of the ideal  $I$ . Since  $\mathrm{gr}(I_1 + I_2) \supseteq \mathrm{gr}I_1 + \mathrm{gr}I_2$ , if  $I(\mathcal{N}_w) = \mathrm{gr}I(\mathcal{B}_{w,t})$ , we obtain a natural map from  $A(\mathcal{N}_w \cap \mathfrak{h})$  onto  $H^*(X_w; \mathbb{C})$ , which turns out to be  $\psi_w$ . Furthermore, we then obtain that  $\psi_w$  is an isomorphism exactly when  $\mathrm{gr}(I(\mathcal{B}_{w,t}) + I(\mathfrak{h})) = I(\mathcal{N}_w) + I(\mathfrak{h})$ .

Some of the results in this paper have been generalized by Peterson [16] to the Kac-Moody setting. Moreover, he has shown that the maps  $\alpha_w$  etc. are all defined over  $\mathbb{Z}$  (instead of  $\mathbb{C}$ ). An account of these results is included in the expository article [9].

The paper is organized as follows. In §2, the basic theorem (Theorem 5) on the zero scheme of the homogeneous principal nilpotent is proven and an example to illustrate the result is given. In §3, we prove the basic result that  $\phi_e$  induces a degree-doubling isomorphism from  $A(\mathcal{N} \cap \mathfrak{h})$  onto  $A(Z_e)$  and in §4 we extend this to the relative case  $\phi_{e,w} : A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow A(Z_e \cap X_w)$ . In §5 we consider a semisimple deformation as a path to an alternate definition of the morphisms  $\psi_w$ . In the Appendix, examples that show the  $\psi_w$  are not always injective are given.

The author would like to thank Dale Peterson for contributing the result in the Appendix. He would also like to thank Hanspeter Kraft and Shrawan Kumar for their comments.

## 2. A description of $I(Z_v)$

**2.1.** The starting point of this paper is the problem of giving a geometric description of the zero scheme of an algebraic vector field on  $X = G/B$  obtained by exponentiating a principal nilpotent  $v \in \mathfrak{g}$ . Such a vector field has its only zero at the unique Borel subgroup of  $G$  whose Lie algebra contains  $v$ . We will suppose  $v \in \mathfrak{b}$ , so the vector field  $V_v$  assigned to  $v$  vanishes only at  $B \in X$ . Recall  $U$  denotes the big open cell in  $X$  centered at  $B$ . Then the zero scheme  $Z_v$  of  $V_v$  is the affine punctual scheme

in  $U$  supported at  $B$  associated to the ideal  $I(Z_v)$  generated by the functions  $V_v(f)$ , where  $f \in A(U)$ , the affine coordinate ring of  $U$ , and  $V_v$  is viewed as a derivation of  $A(U)$ . By definition, the coordinate ring of  $Z_v$  is  $A(Z_v) := A(U)/I(Z_v)$ . We will solve our problem in  $A(U^-)$ , where  $U^-$  is the unipotent radical of  $B^-$ , using the isomorphism  $\rho: U^- \rightarrow U$  defined by  $\rho(u) = uB$ . Let  $\Omega = U^-B$  be the corresponding big cell in  $G$  and  $\Pi: \Omega \rightarrow U^-$  the canonical map defined by the composition

$$\Omega \xrightarrow{m^{-1}} U^- \times B \xrightarrow{\pi_1} U^-$$

where  $m(u, b) = ub$  and  $\pi_1(u, b) = u$ .

**Theorem 4.** *Let  $v \in \mathfrak{b}$  be a principal nilpotent. Then  $\rho^*(I(Z_v)) \subset A(U^-)$  is generated by the components (with respect to any basis) of the  $u^-$ -valued map  $u \mapsto \Pi_* \text{Ad}(u^{-1})v$  where  $\Pi_*$  denotes the differential of  $\Pi$  at the identity  $1_G$  of  $G$ .*

*Proof.* For  $y \in \mathfrak{g}$ , let  $W_y$  be the corresponding right invariant vector field on  $G$ . Thus

$$W_y(g) = R_{g^{-1}}y = \frac{d}{dt}(\exp(ty)g)|_{t=0}.$$

The holomorphic vector field  $V_y$  on  $X$  induced by  $W_y$  is

$$V_y(gB) = \frac{d}{dt}(\exp(ty)gB)|_{t=0}.$$

Let  $\widetilde{W}_y$  be the holomorphic vector field on  $U^-$  defined at  $u \in U^-$  by  $\widetilde{W}_y(u) = \Pi_*(W_y(u))$  where  $\Pi_*: T'_u\Omega \rightarrow T'_uU^-$  stands for the holomorphic differential of  $\Pi$  on the holomorphic tangent space  $T'_u\Omega$  to  $\Omega$  at  $u$ . Let  $\mu: \Omega \rightarrow U$  be the quotient map  $\mu(g) = gB$ . Then  $\rho\Pi = \mu$ , and consequently

$$V_y = \mu_*W_y = (\rho\Pi)_*W_y = \rho_*(\Pi_*W_y) = \rho_*\widetilde{W}_y,$$

where  $\mu_*$  and  $\rho_*$  are analogous to  $\Pi_*$ . Now let  $y = v$  and define  $I_v \subset A(U^-)$  to be the ideal generated by all  $\widetilde{W}_v(f)$  where  $f \in A(U^-)$ .

**Lemma 1.**  *$\rho^*(I(Z_v)) = I_v$ , and  $\rho^*$  induces an isomorphism  $\tilde{\rho}: A(Z_v) \rightarrow A_v := A(U^-)/I_v$ .*

*Proof.* This is obvious from (2.1).

We now compute the ideal  $I_v$ . Note that for  $u \in U^-$ ,

$$W_v(u) = \frac{d}{dt}(uu^{-1}(\exp tv)u)|_{t=0} = L_{u^{-1}}\text{Ad}(u^{-1})v,$$

where  $L_u(g) = ug$ . Since  $\Pi L_u = L_u\Pi$  for  $u \in U^-$ ,

$$\widetilde{W}_v(u) = \Pi_* W_v(u) = \Pi_* L_{u_*} \text{Ad}(u^{-1})v = L_{u_*} \Pi_* \text{Ad}(u^{-1})v.$$

Thus,  $L_{u^{-1}*}(\widetilde{W}_v(u)) = \Pi_* \text{Ad}(u^{-1})v$ . Now suppose that  $v_1, v_2, \dots, v_k$  form a basis of left invariant vector fields on  $U^-$  and write  $\widetilde{W}_v = \sum_{i=1}^k a_i v_i$ , where  $a_1, \dots, a_k$  are in  $A(U^-)$ . Then

$$L_{u^{-1}*}(\widetilde{W}_v(u)) = \sum_{i=1}^k a_i(u) L_{u^{-1}*}(v_i(u)) = \sum_{i=1}^k a_i(u) v_i(1_G).$$

Hence  $a_1, \dots, a_k$  are the components with respect to  $v_1(1_G), \dots, v_k(1_G)$  of  $\Pi_* \text{Ad}(u^{-1})v$ . Since  $a_1, \dots, a_k$  generate  $I_v$ , the theorem is proved.

**2.2.** We now bring in the homogeneous principal nilpotent  $e$ . Let  $\Phi \subset \mathfrak{h}^*$  be the root system of  $(\mathfrak{g}, \mathfrak{h})$  and let  $\Phi^+$  be the set of positive roots, i.e., the roots of  $(\mathfrak{b}, \mathfrak{h})$ . Denote the set of simple roots in  $\Phi^+$  by  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  and choose  $e_i \in \mathfrak{g}_{\alpha_i} \setminus 0$ , where  $\mathfrak{g}_\alpha \subset \mathfrak{g}$  is the root subspace corresponding to each  $\alpha \in \Phi$ . We set  $e = e_1 + \dots + e_l$ . Recall  $\phi_e(u) = \text{Ad}(u^{-1})e$  and note  $\phi_e(u) \in e + \mathfrak{h} + u^-$ . Thus

$$(2.2) \quad \phi_e(u) = e + k_e(u) + \sum_{\alpha > 0} v_{-\alpha}(u) e_{-\alpha},$$

where every  $e_{-\alpha} \in \mathfrak{g}_{-\alpha} \setminus 0$  and  $k_e(u) \in \mathfrak{h}$ . Hence  $I_e$  is defined by the condition  $\phi_e(u) \in e + \mathfrak{h}$ .

Recall that  $e$  induces gradings on  $A(U)$  and  $A(U^-)$  [13]. We now show that  $I(Z_e)$  and  $I_e$  are homogeneous and that  $\rho^*$  determines the graded isomorphism. Let  $s \in \mathfrak{h}$  be the unique element such that  $\langle \alpha_i, s \rangle = 2$  if  $1 \leq i \leq l$ , and  $\gamma: \mathbb{C}^* \rightarrow G$  the one-parameter group such that  $\gamma'(1) = s$ . Since  $U^-$  is  $H$ -invariant,  $\gamma$  determines a  $\mathbb{C}^*$ -action  $\mathbb{C}^* \times U^- \rightarrow U^-$  via  $(t, u) \mapsto t \cdot u = \gamma(t)u\gamma(t)^{-1}$  and this determines the grading of  $A(U^-)$  by setting  $A(U^-)_k := \{f \in A(U^-) \mid t \cdot f = t^k f \text{ for all } t \in \mathbb{C}^*\}$ . Likewise,  $A(U)$  has a grading—namely the one associated with the  $\mathbb{C}^*$ -action  $(t, gB) \rightarrow \gamma(t)gB$  on  $U$ . Clearly,  $\rho$  is  $\gamma$ -equivariant so  $\rho^*$  is a graded isomorphism, and the homogeneity of  $I_e = \rho^*(I(Z_e))$  and  $I(Z_e)$  are equivalent. Recall the height of  $\alpha \in \Phi^+$  is  $\text{ht}(\alpha) = \frac{1}{2} \langle \alpha, s \rangle$ .

**Lemma 2.** *Each  $v_{-\alpha}$  ( $\alpha > 0$ ) is homogeneous of degree  $2(1 + \text{ht}(\alpha))$ , and the map  $k_e$  is homogeneous of degree 2.*

*Proof.* We must show that  $t \cdot v_{-\alpha} = t^{2(1+\text{ht}(\alpha))} v_{-\alpha}$  where as usual  $t \cdot v_{-\alpha}(u) = v_{-\alpha}(t^{-1} \cdot u)$ . Now

$$\begin{aligned}
 t \cdot \phi_e(u) &= \phi_e(\gamma(t)^{-1} u \gamma(t)) = \text{Ad}(\gamma(t)^{-1}) u^{-1} \gamma(t) e \\
 &= \text{Ad}(\gamma(t)^{-1}) \text{Ad}(u^{-1}) \text{Ad}(\gamma(t)) e \\
 &= t^2 \text{Ad}(\gamma(t))^{-1} \text{Ad}(u^{-1}) e \\
 &= t^2 \text{Ad}(\gamma(t))^{-1} \left( e + k_e(u) + \sum v_{-\alpha}(u) e_{-\alpha} \right) \\
 &= e + t^2 \left( k_e(u) + \sum v_{-\alpha}(u) t^{(\alpha, s)} e_{-\alpha} \right) \\
 &= e + t^2 k_e(u) + \sum t^{2(1+\text{ht}(\alpha))} v_{-\alpha}(u) e_{-\alpha}.
 \end{aligned}$$

This establishes the lemma. q.e.d.

To summarize, we state

**Theorem 5.** *Let  $e$  be the principal homogeneous nilpotent  $e = e_1 + \dots + e_l$ . Then the ideals  $I_e$  and  $I(Z_e)$  are homogeneous in  $A(U^-)$  and  $A(U)$  respectively, and  $\tilde{\rho}: A(Z_e) \rightarrow A_e$  is an isomorphism of graded  $\mathbb{C}$ -algebras. Consequently,  $A_e \cong H^*(X; \mathbb{C})$ . If  $\phi_e: U^- \rightarrow \mathfrak{g}$  is the embedding  $\phi_e(u) = \text{Ad}(u^{-1})e$ , then  $I_e$  is generated by the functions  $v_{-\alpha} = \pi_{-\alpha} \phi_e$  ( $\alpha > 0$ ), where  $\pi_{-\alpha}: \mathfrak{g} \rightarrow \mathfrak{g}_{-\alpha} \cong \mathbb{C}$  is the canonical projection.*

**Example.** Let

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_1 & 1 & 0 & 0 \\ u_2 & u_4 & 1 & 0 \\ u_3 & u_5 & u_6 & 1 \end{pmatrix}$$

denote an arbitrary element of  $U^-$ . We have

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so

$$(2.3) \quad \text{Ad}(u^{-1}) \cdot e = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

where

$$\begin{aligned}
 a_{11} &= u_1, & a_{22} &= u_4 - u_1, & a_{33} &= u_6 - u_4, & a_{44} &= -u_6, \\
 a_{21} &= u_2 - u_1^2, & a_{31} &= u_3 - u_2u_4 + u_1(u_1u_4 - u_2), \\
 a_{41} &= -u_3u_6 + u_2(u_4u_6 - u_5) + u_1(-u_3 + u_1u_5 + u_2u_6 - u_1u_4u_6), \\
 a_{32} &= u_5 - u_2 + u_4(u_1 - u_4), \\
 a_{42} &= -u_3 + u_6(u_2 - u_5) + (u_1 - u_4)(u_5 - u_4u_6), \\
 a_{43} &= -u_5 + u_6(u_4 - u_6).
 \end{aligned}$$

The  $a_{ij}$  with  $i > j$  generate  $I_e$ . On the other hand,  $I_e$  is also generated by the coefficients of the vector field

$$\widetilde{W}_e = \sum_{i=1}^6 b_i \frac{\partial}{\partial u_i},$$

which are given as follows:

$$\begin{aligned}
 b_1 &= u_2 - u_1^2, & b_2 &= u_3 - u_1u_2, & b_3 &= -u_1u_3, \\
 b_4 &= u_5 - u_2 + u_4(u_1 - u_4), & b_5 &= -u_3 + u_5(u_1 - u_4), \\
 b_6 &= -u_5 + u_6(u_4 - u_6).
 \end{aligned}$$

Note that the coefficients  $b_1, b_4, b_6$  of  $\widetilde{W}_e$  are matrix entries in (2.3).  $b_1, \dots, b_6$  give a simpler but theoretically less interesting set of generators of  $I_e$  than the entries in (2.3).

### 3. The fundamental isomorphism $\tilde{\phi}_e$

3.1. In this section we show

**Theorem 6.**  $\phi_e$  induces a degree-doubling isomorphism of graded algebras

$$\tilde{\phi}_e: A(\mathcal{N} \cap \mathfrak{h}) \rightarrow A_e.$$

*Proof.* We first show  $\phi_e$  induces a surjective homomorphism  $\phi_e^\#: A(\mathcal{N} \cap (e + \mathfrak{h})) \rightarrow A_e$ . By Theorem 5, it suffices to show  $\phi_e^*$  is surjective. This follows easily from the result of Kostant [13] that if  $\{e, s, f\}$  is an  $\mathfrak{sl}_2$ -triplet, then the map  $U^- \times \mathfrak{g}^f \rightarrow \mathfrak{g}$  sending  $(u, x)$  to  $\text{Ad}(u)(x + f)$  is an isomorphism onto  $e + \mathfrak{h} + u^-$ . Here  $\mathfrak{g}^f$  denotes the centralizer of  $f$ . Notice that  $A(\mathcal{N} \cap (e + \mathfrak{h}))$  is not graded; however the usual grading of  $A(\mathfrak{g}) = \mathbb{C}[z_\alpha, z_{-\beta}, x_i \mid \alpha, \beta > 0, 1 \leq i \leq l]$  ( $\{x_i, z_\alpha, z_{-\beta}\}$  denoting the usual dual basis of  $\mathfrak{g}^*$ ) defines a filtration  $F_0 \subseteq F_1 \subseteq \dots \subseteq F_i \subseteq F_{i+1} \subseteq \dots$  of  $A(\mathcal{N} \cap (e + \mathfrak{h}))$  such that  $F_i F_j \subseteq F_{i+j}$ , where  $F_i := \text{Im } \bigoplus_{j \leq i} A(\mathfrak{g})_j$ .

$A_e$ , being graded, also has a filtration  $F'_i \subseteq F'_{i+1}$  such that  $F'_i F'_j \subseteq F'_{i+j}$ . Since  $\phi_e^*$  is not graded, we must show that for all  $m \geq 0$ ,  $\phi_e^\#(F_m) \subseteq F'_{2m}$ . A typical monomial  $M$  in  $A(\mathfrak{g})$  of degree at most  $m$  has the form

$$M = \prod_{\alpha \in \Delta} (z_\alpha)^{i_\alpha} \prod_{j=1}^l (x_j)^{r_j},$$

where  $\sum i_\alpha + \sum r_j \leq m$ , and all  $i_\alpha$  and  $r_j$  are  $\geq 0$ . Then  $\phi_e^*(M) = 0$  if  $i_\alpha > 0$  for some  $\alpha$  such that  $\text{ht}(\alpha) > 1$ , and  $\phi_e^*(M) \in I_e$  if  $i_\alpha > 0$  for some  $\alpha < 0$ . Hence we may assume

$$M = \prod_{i=1}^l (z_{\alpha_i})^{i_{\alpha_i}} \prod_{j=1}^l (x_j)^{r_j},$$

and then

$$\phi_e^*(M) = k_e^* \left( \prod_{i=1}^l (x_j)^{r_j} \right),$$

which has degree  $2 \sum r_j \leq 2m$ . Therefore  $\phi_e^\#(F_m) \subseteq F'_{2m}$  as claimed. Let  $\text{Gr } A(\mathcal{N} \cap (e + \mathfrak{h})) := F_0 + \sum F_i/F_{i-1}$  be the graded ring associated with the filtration  $F$ . Thus  $\phi_e^\#$  induces

$$\text{Gr } \phi_e^\#: \text{Gr } A(\mathcal{N} \cap (e + \mathfrak{h})) \rightarrow \text{Gr } A_e = A_e.$$

The final step in the proof is to define  $\tilde{\phi}_e$ . By [14, p. 134], there exists a canonical isomorphism

$$(3.1) \quad j: A(\mathfrak{g})/\text{gr}(I(\mathcal{N}) + I(e + \mathfrak{h})) \rightarrow \text{Gr } A(\mathcal{N} \cap (e + \mathfrak{h}))$$

such that if  $f \in A(\mathfrak{g})$  has degree  $\leq p$  and residue class  $\tilde{f}$ , then  $j(\tilde{f})$  is the element of  $F_p/F_{p-1}$  determined by  $f$ . Now  $\text{gr } I(\mathcal{N}) + \text{gr } I(e + \mathfrak{h})$  is clearly  $I(\mathcal{N}) + I(\mathfrak{h})$ , so we can define  $\tilde{\phi}_e$  by the composition

$$(3.2) \quad \begin{aligned} A(\mathcal{N} \cap \mathfrak{h}) &\rightarrow A(\mathfrak{g})/\text{gr}(I(\mathcal{N}) + I(e + \mathfrak{h})) \\ &\rightarrow \text{Gr } A(\mathcal{N} \cap (e + \mathfrak{h})) \rightarrow A_e. \end{aligned}$$

Since  $\dim_{\mathbb{C}} A(\mathcal{N} \cap \mathfrak{h}) = \dim_{\mathbb{C}} A_e$ ,  $\tilde{\phi}_e$  is as claimed and the theorem is proved. q.e.d.

Consequently, we have an isomorphism

$$\tilde{\nu} \tilde{\phi}_e: A(\mathcal{N} \cap \mathfrak{h}) \rightarrow A(Z_e),$$

where  $\tilde{\nu}$  is the inverse of  $\tilde{\rho}$ .

**3.2.** Let  $I_+^W \subset A(\mathfrak{h})$  be the ideal generated by the homogeneous  $W$ -invariants. The inclusion map  $i: \mathfrak{h} \rightarrow \mathfrak{g}$  induces an isomorphism  $\tilde{i}$ :



$A(\mathcal{N} \cap \mathfrak{h}) \rightarrow S_W := A(\mathfrak{h})/I_+^W$ . In [1], it was shown that if  $\pi$  denotes the projection of  $\mathfrak{g}$  onto  $\mathfrak{h}$ , then the map  $\tau_e: \mathfrak{u}^- \rightarrow \mathfrak{h}$  given by  $\tau_e(n) = \pi[e, n]$  induces an isomorphism of graded algebras

$$(3.3) \quad \tilde{\tau}_e: S_W \rightarrow A(\mathfrak{u}^-)/\exp^*(I_e).$$

It seems worthwhile to use the above results to reprove this.

**Corollary.**  $\tilde{\tau}_e$  is a degree-doubling isomorphism of graded algebras. If  $\omega_1, \dots, \omega_l$  are the fundamental dominant weights of  $\mathfrak{h}$  with respect to  $\alpha_1, \dots, \alpha_l$ , then  $\tilde{\tau}_e(\omega_i) = \bar{z}_{-i}$  if  $1 \leq i \leq l$ , where "bars" denote residue classes.

*Proof.* In fact,  $\tau_e = \tau\phi_e \exp = k_e \exp$ . We show  $\pi^*$  induces the inverse of  $\tilde{\tau}_e$ . To see that  $\pi^*(I_+^W) \subseteq I(\mathcal{N}) + I(\mathfrak{h})$ , use the fact that for any  $f \in A(\mathfrak{h})^W$ , there exists a  $g \in A(\mathfrak{g})^G$  such that  $i^*g = f$ . Then  $\pi^*f - g$  vanishes on  $\mathfrak{h}$ ; hence  $\pi$  induces a morphism  $\tilde{\pi}: S_W \rightarrow A(\mathcal{N} \cap \mathfrak{h})$  which is the inverse of  $\tilde{\tau}_e$ . Thus  $\tilde{\tau}_e$  is an isomorphism. Next, let  $a = \sum a_\alpha e_{-\alpha} \in \mathfrak{u}^-$ . We may suppose  $\{e_i, a_i^\vee, e_{-i}\}$  forms an  $\mathfrak{sl}_2$ -triplet. Hence

$$[e, a] = \sum a_{\alpha_j} [e_j, e_{-j}] = \sum a_{\alpha_j} \alpha_j^\vee.$$

Now for any fundamental dominant weight  $\omega_i$ ,

$$\tau_e^*(\omega_i)(a) = \omega_i(\tau_e(a)) = \sum a_{\alpha_j} \omega_i(h_j) = a_{\alpha_j}.$$

This shows that  $\tau_e^*(\omega_i) = z_{-i}$ , and completes the proof.

**3.3.** To summarize the maps that have been introduced, we note the following commutative diagram of isomorphisms:

$$(3.4) \quad \begin{array}{ccccc} A(\mathcal{N} \cap \mathfrak{h}) & \xrightarrow{\tilde{i}} & & & S_W \\ \tilde{\phi}_e \downarrow & & A(\mathfrak{u}^-)/\exp^* I_e & \xrightarrow{\tilde{\tau}_e} & \downarrow \beta \\ & \swarrow \tilde{\exp} & & & H(X; \mathbb{C}) \\ A_e & \xrightarrow{\alpha\tilde{\rho}} & & & \end{array}$$

It is well known that as a  $W$ -module,  $S_W$  is the regular representation. There is no obvious  $W$ -module structure on  $A_e$  however. Recently, Dale Peterson showed that there exists an action of  $W$  on  $U^-$  such that the functions  $v_{-\alpha}$  of (2.2) are a fundamental system of generators for  $A(U^-)^W$ , which is a polynomial ring. Moreover,  $\tilde{\phi}_e \tilde{\pi}: S_W \rightarrow A_e$  is  $W$ -equivariant.

**4. The fundamental isomorphisms  $\phi_{e,w}$**

**4.1.** In this section we prove Theorem 2. The main step is to show that  $\phi_e$  induces an isomorphism between  $A(\mathcal{N}_w \cap \mathfrak{h})$  and

$$A_{e,w} := A(U^-)/I_e + I(U^- \cap \overline{BwB}).$$

First we will show that  $\tilde{\rho}: A(Z_e) \rightarrow A_e$  induces an isomorphism  $\tilde{\rho}_w$  from  $A(Z_e \cap X_w)$  onto  $A_{e,w}$ . Note that  $\rho$  induces an isomorphism of varieties between  $U^- \cap \overline{BwB}$  and  $U \cap X_w$ , hence an isomorphism  $\rho_w^*: A(U \cap X_w) \rightarrow A(U^- \cap \overline{BwB})$ . Thus, by the definition of  $I_e$ , we obtain the isomorphism  $\tilde{\rho}_w: A(U \cap X_w)/I(Z_e) = A(Z_e \cap X_w) \rightarrow A(U^- \cap \overline{BwB})/I_e = A_{e,w}$ .

Recall that  $\mathcal{N}_w$  is by definition  $\text{Ad}(Bw^{-1}B)e$ . Put  $u \cdot e = \phi_e(u)$ .

**Lemma 3.**

- (1)  $\phi_e^{-1}(\mathcal{N}_w) = U^- \cap \overline{BwB}$  as varieties.
- (2)  $I(U^- \cap \overline{BwB}) \subset A(U^-)$  is homogeneous.

*Proof.* For (1), suppose  $u \cdot e = b_1 w b_2 \cdot e$  for some  $u$  in  $U^-$  and  $b_1, b_2$  in  $B$ . Since the centralizer  $G_e$  of  $e$  is contained in  $B$ ,  $u = b'_1 w b'_2$  for some  $b'_1, b'_2$  in  $B$ . Thus  $u \in U^- \cap \overline{BwB}$  so  $\phi_e^{-1}(\mathcal{N}_w) = U^- \cap \overline{BwB}$ . That  $\phi_e$  is an isomorphism follows from the fact that  $\phi_e$  is a closed immersion. For (2) note that  $\overline{BwB}$  is stable under the 1-p.s.g.  $\gamma$ . Using this one proves the homogeneity in the same way it is established for the standard action in [15, p. 21]. This proves the lemma.

It follows that  $\phi_e^*(I(\mathcal{N}_w)) = I(U^- \cap \overline{BwB})$  so we obtain a surjective  $\mathbb{C}$ -algebra homomorphism

$$\phi_{e,w}^\# : A(\mathcal{N}_w \cap (e + \mathfrak{h})) \rightarrow A_{e,w}.$$

Arguing as in (3.1) and (3.2), using the fact that  $I(\mathcal{N}_w)$  is homogeneous, we obtain a morphism of graded rings

$$A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow \text{Gr } A(\mathcal{N}_w \cap (e + \mathfrak{h}))$$

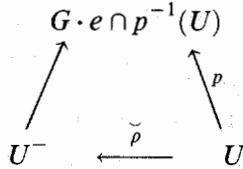
which composed with  $\text{Gr } \phi_{e,w}^\#$  yields a morphism

$$\gamma_w : A(\mathcal{N}_w \cap \mathfrak{h}) \rightarrow \text{Gr } A_{e,w}.$$

However, both ideals  $I_e$  and  $I(U^- \cap \overline{BwB})$  being homogeneous, it is trivial that  $\text{gr}(I_e + I(U^- \cap \overline{BwB})) = I_e + I(U^- \cap \overline{BwB})$ , so  $\text{Gr } A_{e,w} \cong A_{e,w}$  and we finally obtain  $\tilde{\phi}_w$  as the composition of surjective morphisms

$$A(\mathcal{N}_w \cap \mathfrak{h}) \xrightarrow{\gamma_w} A_{e,w} \xrightarrow{\tilde{\rho}_w^{-1}} A(Z_e \cap X_w).$$

In order to show that  $\tilde{\phi}_w$  is an isomorphism, we will show that  $\gamma_w$  is an isomorphism. Note first that  $\dim_{\mathbb{C}} A(\mathcal{N}_w \cap \mathfrak{h}) \geq \dim_{\mathbb{C}} A_{e,w}$ . Let  $p: G \cdot e \rightarrow X$  be the projection defined by  $p(g \cdot e) = gB$ . Then we have the commutative diagram



where  $\tilde{p}(uB) = u^{-1}$ . Since  $\mathcal{N}$  is normal and  $G \cdot e$  has even codimension in  $\mathcal{N} = \overline{G \cdot e}$ , one knows that  $A(G \cdot e \cap p^{-1}(U)) = A(\mathcal{N} \cap p^{-1}(U))$ . Therefore, since  $p^{-1}(U)$  is a Zariski open neighborhood of  $e$  in  $G \cdot e$ ,  $\dim_{\mathbb{C}} A(\mathcal{N}_w \cap (e + \mathfrak{h})) = \dim_{\mathbb{C}} A(p^{-1}(U) \cap \mathcal{N}_w \cap (e + \mathfrak{h}))$  for all  $w$  in  $W$ . Now,  $p$  induces a morphism  $p_w^\#$  sending  $\tilde{p}^{-1}(A_{e,w})$  onto  $A(p^{-1}(U) \cap \mathcal{N}_w \cap (e + \mathfrak{h}))$ , the surjectivity holding for each  $w$  since it holds for the longest element  $w_0$ . Thus

$$\dim_{\mathbb{C}} A_{e,w} = \dim_{\mathbb{C}} \tilde{p}^{-1}(A_{e,w}) \geq \dim_{\mathbb{C}} A(\mathcal{N}_w \cap (e + \mathfrak{h})).$$

To finish the proof that  $\gamma_w$  is an isomorphism, it will be sufficient to note that  $\text{Gr } A(\mathcal{N}_w \cap (e + \mathfrak{h})) \cong A(\mathcal{N}_w \cap \mathfrak{h})$ . Hence it suffices to check that

$$(4.1) \quad \text{gr}(I(\mathcal{N}_w) + I(e + \mathfrak{h})) = I(\mathcal{N}_w) + I(\mathfrak{h}).$$

This is clear, however, since  $I(e + \mathfrak{h})$  is generated by linear functions and  $I(\mathcal{N}_w)$  is homogeneous. This completes the proof of Theorem 2 except checking commutativity, which is left to the reader.

**4.2.** We now prove Theorem 1. Since  $V_e$  is tangent to the set of regular points of  $X_w$ , it follows from [3, Theorem 5] that there exists a filtration of  $A(Z_e)$  such that there is a commutative diagram

$$(4.2) \quad \begin{array}{ccc}
 \text{Gr } A(Z_e) & \xrightarrow{\alpha} & H^*(X; \mathbb{C}) \\
 \downarrow & & \downarrow \\
 \text{Gr } A(Z_e \cap X_w) & \xrightarrow{\alpha_w} & H^*(X_w; \mathbb{C})
 \end{array}$$

where  $\alpha$  is an isomorphism of graded  $\mathbb{C}$ -algebras, the vertical maps are the restrictions, and the filtration on  $A(Z_e \cap X_w)$  defining  $\text{Gr } A(Z_e \cap X_w)$  is the image of the filtration on  $A(Z_e)$ . By [1],  $A(Z_e)$  is graded and  $\text{Gr } A(Z_e) \cong A(Z_e)$ . Similarly, by naturality,  $A(Z_e \cap X_w)$  is graded (by §2) and isomorphic with  $\text{Gr } A(Z_e \cap X_w)$ . Combining this with Theorem 2 finishes the proof of Theorem 1.

### 5. The semisimple deformation

**5.1.** In this section we will seek a method for defining the morphisms  $\psi_w$  without factoring through  $A(Z_e \cap X_w)$ . The method is motivated by the deformation argument employed in [8] and the computation of  $H^*(X_w; \mathbb{C})$  in terms of the orbit  $W \cdot t$  of a regular  $t \in \mathfrak{h}$ . We first recall that computation. Put  $W(w) = \{v \in W \mid v \leq w\}$ , where  $<$  is the partial order on  $W$  associated to  $B$ .

**Theorem 7** [3, Theorem 4]. *Let  $t \in \mathfrak{h}$  be regular. For each  $w \in W$ , there exists a degree-doubling isomorphism  $\lambda_w: A(\mathfrak{h})/\text{gr } I_{\mathfrak{h}}(W(w^{-1}) \cdot t) \rightarrow H^*(X_w; \mathbb{C})$  such that if  $X_v \subseteq X_w$ , the diagram*

$$\begin{array}{ccc} A(\mathfrak{h})/\text{gr } I_{\mathfrak{h}}(W(w^{-1}) \cdot t) & \xrightarrow{\lambda_w} & H^*(X_w; \mathbb{C}) \\ \downarrow & & \downarrow \\ A(\mathfrak{h})/\text{gr } I_{\mathfrak{h}}(W(v^{-1}) \cdot t) & \xrightarrow{\lambda_v} & H^*(X_v; \mathbb{C}) \end{array}$$

*commutes, where the vertical maps are the natural restrictions, and  $I_{\mathfrak{h}}(W(w) \cdot t)$  denotes the ideal of  $W(w) \cdot t$  in  $A(\mathfrak{h})$ .*

Suppose  $V$  is an affine variety in  $\mathfrak{g}$  with defining ideal  $I \subset A(\mathfrak{g})$ . The associated cone  $K(V)$  is by definition the affine variety in  $\mathfrak{g}$  determined by the homogeneous ideal  $\text{gr } I$ . A basic result of Borho and Kraft [8] says that  $\text{gr } I(G \cdot t) = I(\mathcal{N})$  for any regular semisimple element  $t$  of  $\mathfrak{g}$ . We will use this to determine the ideals  $I(\mathcal{N}_w)$  in a useful way. Recall that  $s$  is the unique regular element of  $\mathfrak{h}$  characterized by the condition  $\langle \alpha_i, s \rangle = 2$  for  $i = 1, \dots, l$ . For each  $w \in W$ , let  $B_{w,s}$  denote the Zariski closure of  $Bw^{-1}B \cdot s := \text{Ad}(Bw^{-1}B)s$  in  $\mathfrak{g}$ . We will prove

**Theorem 8.** *Let  $w$  be an arbitrary element of  $W$ . Then the following hold:*

(1)  $B_{w,s}$  is a normal, irreducible subvariety of  $\mathfrak{g}$  of dimension  $l(w) + \dim_{\mathbb{C}} X$ . In addition,  $B_{w^{-1},s}$  is smooth if  $X_w$  is.

(2)  $\mathcal{N}_w$  is an irreducible component of the associated cone  $K(B_{w,s})$ . In particular,  $\text{gr } I(B_{w,s}) \subseteq I(\mathcal{N}_w)$ .

(3)  $I(W(w^{-1}) \cdot s) = I(B_{w,s}) + I(\mathfrak{h})$ , where  $I(W(w) \cdot s)$  is the ideal in  $A(\mathfrak{g})$  of functions vanishing on  $W(w) \cdot s$ .

We obtain therefore a surjective degree-doubling algebra homomorphism

$$\nu_w: A(\mathfrak{g})/\text{gr } I(B_{w,s}) + I(\mathfrak{h}) \rightarrow H^*(X_w; \mathbb{C})$$

as the composition:

$$\begin{aligned}
 & A(\mathfrak{g})/\text{gr } I(B_{w,s}) + I(\mathfrak{h}) \xrightarrow{\text{nat. map}} A(\mathfrak{g})/\text{gr}(I(B_{w,s}) + I(\mathfrak{h})) \\
 & = A(\mathfrak{g})/\text{gr } I(W(w^{-1}) \cdot s) \xrightarrow{i^*} A(\mathfrak{h})/\text{gr } I_{\mathfrak{h}}(W(w^{-1}) \cdot s) \xrightarrow{\lambda_w} H^*(X_w; \mathbb{C}).
 \end{aligned}$$

Here we have used the fact that  $i^*(\text{gr } I(W(w) \cdot s)) \subseteq \text{gr } I_{\mathfrak{h}}(W(w) \cdot s)$ . Clearly,  $i^*$  is an isomorphism, so  $\nu_w$  is an isomorphism precisely when

$$(5.1) \quad \text{gr}(I(B_{w,s}) + I(\mathfrak{h})) = \text{gr } I(B_{w,s}) + I(\mathfrak{h}).$$

**Remark.** Notice that unlike (4.1), (5.1) is not automatically true, the difference being that  $I(B_{w,s})$  is neither homogeneous nor generated by linear functions.

Since  $\text{gr } I(B_{w,s}) \subseteq I(\mathcal{N}_w)$ , there is a similar homomorphism

$$\sigma_w: A(\mathfrak{g})/\text{gr } I(B_{w,s}) + I(\mathfrak{h}) \rightarrow A(\mathcal{N}_w \cap \mathfrak{h}).$$

**Corollary.** For each  $w \in W$  the following hold: (1) The following diagram is commutative:

$$(5.2) \quad \begin{array}{ccc} A(\mathfrak{g})/\text{gr } I(B_{w,s}) + I(\mathfrak{h}) & & \\ \sigma_w \swarrow & & \searrow \nu_w \\ A(\mathcal{N}_w \cap \mathfrak{h}) & \xrightarrow{\psi_w} & H^*(X_w; \mathbb{C}). \end{array}$$

(2)  $\nu_w, \psi_w$  and  $\sigma_w$  are all isomorphisms if and only if (5.1) holds.

(3) If  $\text{gr } I(B_{w,s}) + I(\mathfrak{h}) = I(\mathcal{N}_w) + I(\mathfrak{h})$ , then  $\psi_w = \nu_w$ .

The point of (3) is that when  $\text{gr } I(B_{w,s}) = I(\mathcal{N}_w)$ ,  $\psi_w$  can be defined without factoring through  $A(\mathcal{Z}_e \cap X_w)$ . It seems to be an interesting problem to determine when this is so.

Parts (2) and (3) of the corollary follow from the above discussion. We will first prove Theorem 8 and then establish commutativity of (5.2). We begin with a useful fact.

**Lemma 4.** For each  $w \in W$ ,  $B_{w,s} = \overline{Bw^{-1}B \cdot s}$ .

*Proof.* Since  $s$  is regular and semisimple,  $G \cdot s$  is  $G/H$ , where  $H$  is the algebraic torus corresponding to  $\mathfrak{h}$ . The orbit map  $G \rightarrow G \cdot s$  sends closed  $H$ -invariant subsets of  $G$  onto closed subsets of  $G \cdot s$ . In particular,  $\overline{Bw^{-1}B \cdot s}$  is closed in  $G \cdot s$ . Since  $G \cdot s$  is closed, the lemma is established.  $\square$

It follows that  $B_{w,s}$  is isomorphic with  $\overline{Bw^{-1}B/H}$  and is bundle over  $\overline{Bw^{-1}B/B} = X_{w^{-1}}$  with fibre  $B/H$ . All the statements in Theorem 8 (1) follow immediately from this.

Next we will show that  $\mathcal{N}_w$  is an irreducible component of  $K(B_{w,s})$ . In fact, it follows immediately from [14, Satz, p. 133] that

$$K(B_{w,s}) = \overline{\mathbb{C}^* B_{w,s}} \setminus \mathbb{C}^* B_{w,s} = \mathcal{N} \cap \overline{\mathbb{C}^* B_{w,s}}.$$

This implies that  $\mathcal{N}_w \subseteq K(B_{w,s})$ . For  $e \in \mathbb{C}^* B \cdot s$ , since  $te + s$  and  $s$  are conjugate via  $B$  for any  $t \in \mathbb{C}$ , and this implies  $\overline{\mathbb{C}^* B w^{-1} B \cdot s} \subseteq K(B_{w,s})$ . As  $B_{w,s}$  is irreducible, all irreducible components of  $K(B_{w,s})$  have the same dimension, namely  $\dim_{\mathbb{C}} B_{w,s}$  [14, p. 131]. In particular, as  $\mathcal{N}_w$  also has this dimension and is irreducible, it is an irreducible component. We next prove (3). By Lemma 4 and the fact that  $G \cdot s \cap \mathfrak{h} = W \cdot s$ , it follows from  $\overline{BwB} = \bigcup_{x \leq w} BxB$  that  $B_{w,s} \cap \mathfrak{h} = W(w^{-1}) \cdot s$ . Hence it suffices to show that the scheme theoretic intersection  $B_{w,s} \cap \mathfrak{h}$  is reduced. But  $B_{w,s} \cap \mathfrak{h}$  is locally closed in  $G \cdot s \cap \mathfrak{h}$  and  $G \cdot s \cap \mathfrak{h}$  is reduced, so it automatically follows that  $B_{w,s} \cap \mathfrak{h}$  is also reduced. This completes the proof of Theorem 8. We omit the proof of the commutativity of (5.2).

### Appendix (after D. Peterson)

The notation of the Appendix is the same as that introduced in §§1 and 2. In particular  $Z_e \cap X_w$  will refer to the scheme-theoretic intersection of the zero scheme  $Z_e$ , associated to the principal homogeneous nilpotent  $e$  in  $\mathfrak{b}$ , and the Schubert variety  $X_w = \overline{BwB/B}$  in the flag variety  $X = G/B$  of  $G$ . It was conjectured in [1] that the morphisms  $\alpha_w$  of §1, sending the coordinate ring  $A(Z_e \cap X_w)$  onto  $H^*(X_w; \mathbb{C})$ , are isomorphisms. As mentioned in the Introduction, this has now been established when  $G$  is  $SL_n(\mathbb{C})$  [5].

The purpose of this Appendix is to prove the following negative result, which impinges on Theorem 1 of this paper. Recall,  $\{\alpha_1, \dots, \alpha_l\}$  is the set of simple roots and  $\omega_1, \dots, \omega_l$  the corresponding fundamental dominant weights satisfying  $(\omega_i, \alpha_j^\vee) = \delta_{ij}$  where  $(\ , \ )$  is the inner product on  $\mathfrak{h}^*$  and for  $\alpha \in \Phi$ ,  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ .

**Theorem.** *Suppose  $w \in W$  has length  $l(w_0) - 1$  and write  $w = w_0 r_i$ , where  $w_0$  is the longest element of  $W$  and  $r_i$  is the reflection corresponding to a simple root  $\alpha_i$ . If the fundamental dominant weight  $\omega_i$  corresponding to  $\alpha_i$  is not miniscule, then the inclusion  $i_w$  of  $X_w$  into  $X$  induces an isomorphism between the coordinate rings  $A(Z_e)$  and  $A(Z_e \cap X_w)$ . Consequently,  $\dim_{\mathbb{C}} \ker \alpha_w = \#\{v \in W \mid v \not\leq w\}$ .*

Recall that a dominant weight  $\lambda$  is called minuscule if  $(\lambda, \beta^\vee)$  is 0 or 1 for all positive roots  $\beta$ .

From now on,  $G$  will be assumed to be simple. The theorem is based on the

**Proposition.** *Suppose  $w$  is as in the theorem. Then  $I(\overline{BwB} \cap U^-)$  in  $A(U^-)$  is a principal ideal generated say by  $F_w \in A(U^-)$ , and if  $\omega_i$  is not minuscule, then  $F_w$  is in the kernel of the homomorphism from  $A(U^-)$  onto  $H^*(X; \mathbb{C})$  defined in §1.*

Before proving this, let us show how to obtain the main result. Denote by  $\overline{F}_w$  in  $A(Z_e)$  the residue class of  $F_w$ . Since the kernel of the natural map  $i_w$  from  $A(Z_e)$  into  $A(Z_e \cap X_w)$  is the image of  $I(\overline{BwB} \cap U^-)$  in  $A(Z_e)$ , this kernel is generated by  $\overline{F}_w$ . But  $\alpha(\overline{F}_w) = 0$  by the Proposition, and therefore  $\overline{F}_w = 0$ , so  $i_w$  is an isomorphism. Using the long exact sequence of cohomology for the pair  $(X, X_w)$  shows that  $\dim_{\mathbb{C}} \ker \alpha_w = \dim_{\mathbb{C}} H^*(X, X_w; \mathbb{C})$  which is  $\#\{v \in W \mid v \not\leq w\}$  as asserted. This proves the theorem.

We will break the proof of the proposition into a number of steps, the first being the definition of  $F_w$ . Assume  $\omega_j$  is any fundamental dominant weight and  $V_j = L(\omega_j)$  is the irreducible  $G$ -module with highest weight  $\omega_j$ . Let  $V_j^*$  be the dual  $G$ -module. If  $\mu$  is a weight of  $V$  (resp.  $V^*$ ), let  $V_\mu$  (resp.  $V_\mu^*$ ) denote the corresponding  $\mu$ -weight space. Let  $v^+$  and  $v^*$  be highest weight vectors in  $V_i$  and  $V_i^*$  respectively, and define  $F_w \in A(U^-)$  as the lowest weight component of  $u \cdot v^+$ , i.e.,

$$F_w(u) = \langle v^*, u \cdot v^+ \rangle$$

for  $u \in U^-$ . Since the highest weight of  $V_i^*$  is  $-\omega_0(\omega_i)$ ,  $F_w(u)$  is simply the  $\omega_0(\omega_i)$  component of  $u \cdot v^+$ .

The second step in the proof is to show that the variety  $V(F_w)$  of  $F_w$  in  $U^-$  is  $\overline{BwB} \cap U^-$ . The proof given here can easily be modified to show that if  $F_w$  is viewed in  $A(G)$ , then  $V(F_w) = \overline{BwB}$ . Note first that  $\overline{BwB} \cap U^-$  is a Zariski closed subset of  $U^-$  isomorphic with  $X_w \cap U$ , where  $U$  is the dual open cell in  $X$  containing  $B$ . The Schubert decomposition  $\bigcup_{w' \leq w} Bw'B/B$  is a locally closed decomposition of  $X_w$ , and there is a corresponding one  $\bigcup_{w' \leq w} (Bw'B \cap U^-)$  for  $\overline{BwB} \cap U^-$ .

Suppose now that  $u \in V(F_w)$ . Then  $u \in Bw'B$  for some  $w' \in W$ , so we want to show  $w' \leq w$ . Since multiplication by  $w_0$  reverses order and  $w_0^2 = 1$ , it suffices to show  $w_0 w' \geq r_i$ . If not, there exists a reduced expression  $w_0 w' = r_{j_1} \cdots r_{j_k}$  not involving  $r_i$ , so  $w_0 w'(\omega_i) = \omega_i$ , i.e.,

$w'(\omega_i) = w_0(\omega_i)$ . Since  $u \in Bw'B$ ,  $u \cdot v^+ = bn' \cdot v^+$  for some  $b \in B$  and  $n' \in w'H$ , where  $H$  is the maximal torus corresponding to  $\mathfrak{h}$ . This means  $b^{-1}u \cdot v^+$  is a lowest weight vector, which is impossible since, by the definition of  $F_w(u)$ , the lowest weight component of  $u \cdot v^+$  is zero. Hence  $w' \leq w$ , and  $V(F_w) \subseteq \overline{BwB} \cap U^-$ .

To see the opposite inclusion, it suffices to show that  $BwB \cap U^- \subset V(F_w)$  since, as shown above,  $\overline{BwB} \cap U^- = \bigcup_{w' \leq w} Bw'B \cap U^- = \overline{BwB} \cap U^-$ . Thus let  $u \in BwB \cap U^-$ . Then  $u \cdot v^+ \in \sum_{\lambda \geq w(\omega_i)} V_\lambda$ , so  $\langle v^*, u \cdot v^+ \rangle = 0$  since  $w(\omega_i) \neq w_0(\omega_i)$ . (Indeed,  $w_0(\omega_i) = wr_i(\omega_i) = w(\omega_i - \alpha_i)$ .) Consequently  $u \in V(F_w)$  and we have established that  $V(F_w) = \overline{BwB} \cap U^-$ .

In order to show that  $I(\overline{BwB} \cap U^-)$  is  $(F_w)$ , observe that since  $\overline{BwB} \cap U^-$  is irreducible, it suffices to show that  $V(F_w)$  has no multiple components. To establish this, we show that the differential  $dF_w$  is nonzero at all points of  $BwB \cap U^-$ . We begin with a

**Lemma.** *Let  $r_j$  be the simple reflection such that  $w = r_j w_0 = w_0 r_i$ . Then  $w(\omega_i) + r_j(\omega_j) = 0$ .*

*Proof.*  $r_j$  exists since  $w_0 r_i w_0$  has length one. Clearly  $w_0(\alpha_i) = -\alpha_j$ , so  $w(\omega_i) = w_0 r_i(\omega_i) = w_0(\omega_i) + \alpha_j$ . Thus it suffices to show  $w_0(\omega_i) = -\omega_j$ . This follows since  $w_0$  interchanges the positive and negative roots,  $w_0^2 = 1$ , and  $(w_0(\omega_i), \alpha_j^\vee) = (\omega_i, w_0(\alpha_j^\vee)) = -(\omega_i, \alpha_i^\vee) = -1$ . This completes the proof.

Let  $f_j$  be a nonzero element of  $\mathfrak{g}_{-\alpha_j}$ . For  $u \in U^-$ , we have

$$\begin{aligned} \left( \frac{d}{dt} F_w((\exp t f_j)u) \right) \Big|_{t=0} &= \langle v^*, f_j \cdot (u \cdot v^+) \rangle \\ &= -\langle f_j \cdot v^*, u \cdot v^* \rangle. \end{aligned}$$

But, for  $u \in BwB \cap U^-$ , the  $w(\omega_i)$ -component of  $u \cdot v^+$  is nonzero, and, since  $v^*$  is a highest-weight vector of weight  $\omega_j$ ,  $f_j \cdot v^*$  is a nonzero element of the one-dimensional space  $V_{\omega_j - \alpha_j}^* = V_{r_j(\omega_j)}^*$ . Since  $w(\omega_i) + r_j(\omega_j) = 0$ , we deduce that  $\langle f_j \cdot v^*, u \cdot v^+ \rangle \neq 0$ . Hence  $dF_w \neq 0$  at every point of  $BwB \cap U^-$ , so  $V(F_w)$  has no multiple components and we deduce that  $I(\overline{BwB} \cap U^-) = (F_w)$ .

It remains to show that  $\alpha(\overline{F_w}) = 0$  if  $\omega_i$  is not minuscule. Let  $P$  be the stabilizer of the line  $\mathbb{C}v^+$ . Note that  $\dim_{\mathbb{C}} G/P$  is the number of positive roots involving  $\alpha_i$ . There exists a commutative diagram of



**C-algebra homomorphisms**

$$\begin{array}{ccc}
 A(U^-/U^- \cap P) & \xrightarrow{\alpha_p^\vee} & H^*(G/P; \mathbb{C}) \\
 \pi^* \downarrow & & \downarrow \pi^* \\
 A(U^-) & \xrightarrow{\alpha^\vee} & H^*(X; \mathbb{C})
 \end{array}$$

where the vertical maps  $\pi^*$  are induced by the natural projections  $\pi$ ,  $\alpha^\vee$  is the obvious lift of  $\alpha$ , and  $\alpha_p^\vee$  is the lift of the isomorphism

$$\alpha_p: A(U^-/U^- \cap P)/I(Z_e) \rightarrow H^*(G/P; \mathbb{C})$$

established for  $G/P$  in the same way as  $\alpha$  for  $G/B$  ([4] and [1]). Since  $F_w(up) = F_w(u)$  if  $p \in U^- \cap P$ , there exists an element  $G_w$  of  $A(U^-/U^- \cap P)$  of the same degree as  $F_w$  such that  $F_w = \pi^* G_w$ . Since  $\text{deg } F_w = \text{ht}(\omega_i - w_0(\omega_i))$ , it will follow that  $\alpha(F_w) = 0$  if  $\text{ht}(\omega_i - w_0(\omega_i)) > \dim_{\mathbb{C}} G/P$ . Recall that if  $\omega$  is a positive weight, say  $\omega = \sum a_i \alpha_i$ , then  $\text{ht}(\omega) = \sum a_i$ .

**Lemma.** Let  $\rho^\vee = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta^\vee$ . Then for any positive weight  $\lambda$ ,  $\text{ht}(\lambda) = (\lambda, \rho^\vee)$ .

*Proof.* Since  $r_i(\rho^\vee) = \rho^\vee - \alpha_i^\vee$  for  $i = 1, \dots, l$ , it follows that  $(\rho^\vee, \alpha) = 1$  for all simple roots  $\alpha$ . Hence  $(\rho^\vee, \sum a_i \alpha_i) = \sum a_i$ .

Now

$$\begin{aligned}
 \text{deg } G_w &= \text{deg } F_w = (\rho^\vee, \omega_i - w_0(\omega_i)) \\
 &= (\rho^\vee, \omega_i) - (w_0 \rho^\vee, \omega_i) = 2(\rho^\vee, \omega_i) \\
 &\geq \#\{\beta \in \Phi^+ \mid (\beta^\vee, \omega_i) > 0\} = \dim_{\mathbb{C}} G/P
 \end{aligned}$$

with equality if and only if  $(\beta^\vee, \omega_i) = 1$  for all  $\beta$  involving  $\alpha_i$ . Consequently, if  $\omega_i$  is not minuscule, then  $\alpha_p^\vee(G_w) = 0$  which shows  $\alpha(\overline{F}_w) = 0$  and finally completes the proof of the proposition.

**Remark.** The only simple group for which every fundamental dominant weight is minuscule is  $SL_n(\mathbb{C})$ .

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